A Comparative Study on Decomposition of Test Signals Using Variational Mode Decomposition and Wavelets

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Abstract: The decomposition of signals into their primitive or fundamental constituents play a vital role in removing noise or unwanted signals, thereby improving the quality and utility of the signals. There are various decomposition techniques, among which the linear wavelet technique and the Variational Mode Decomposition (VMD) are the most recent and widely used ones. This paper presents a comparative study of the decomposition of spatially inhomogeneous test functions namely Doppler and Bumps used by statisticians. An effort is made in this article to compare the efficiency of the noise removal in the resulting decompositions at various approximation levels using wavelets and by varying the number of reconstruction modes in VMD. Surprisingly it is found that the VMD technique yields better results with more accuracy for a specific set of parameters irrespective of the spatial character of the function.

Keywords: Variational mode decomposition, Doppler, Bumps.

1. Introduction

Myriads of techniques exist, for a signal to be processed and analyzed. Decomposition is one such technique, which includes determining the number of signals present, their epochs and amplitudes. The key idea of analysis is to represent signals as a superposition of simpler signals so that each such signal can be operated upon independently. The motivation for our work arose from the investigations of brain waves such as electroencephalogram and visual evoked responses, the desire to distinguish between multiple echoes in a noisy environment in radar and sonar and the need to locate the seismic source in seismology where the data consists of the arrivals of various waves. Under all such occasions, there has been a requirement to seek methods by which composite waveforms may be decomposed[1]. However, the digital data decomposition involves plethora of problems like filter realizability, signal resolution capability, the effects of additive noise and a frequency (spectrum) compatibility between signal waveform and filter response pulse. Such problems can be dealt by implementing various linear filter operations and decomposition techniques.

For signals that are to be processed, the prior knowledge about the characteristics of the signal is either known or unknown. Even when the knowledge about the signal is known, information about its spatial homogeneity may still be unknown. Spatially inhomogeneous functions may have, for example, jump discontinuities and high frequency oscillations [2]. Such functions might require a greater amount of smoothing in some portions of the domain, and less smoothing in other places [3].However, various techniques exist for signal processing to be carried out with or without prior details. Our effort in this paper is to identify a decomposition technique that would yield efficient denoising results inspite of the absence of prior knowledge or spatial homogeneity. For this purpose, we make a comparison between the widely used linear wavelet techniques and the recently developed Variational Mode Decomposition(VMD) [4] by employing them on the standard test functions, Bumps and Doppler defined by Donoho & Johnstone (1994) . These functions have been particularly chosen because they caricature spatially variable functions arising in imaging, spectroscopy and other scientific signal processing [5]. Similar signal decomposition techniques that are

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based on linear wavelets have been used for ECG signals [6], for smoothing noisy data [7-8] and for curve estimation [9]. We present this study on decomposition of spatially inhomogeneous signals using VMD and linear wavelet techniques to compare the efficiency of noise removal that has not been attempted so far.
The rest of the paper is organized as follows. Section 2 briefly reviews about wavelets and VMD. The comparison between decomposition techniques discussed above is explained in Section 3. The observations made in our study is summarized and concluded in Section 4.

2. Review of Wavelet and VMD

A. Wavelet

In this sub-section we give a brief review of wavelet theory, on which the function representation is to be made. Wavelet decomposition method is motivated by the fact that the important information appears through a simultaneous analysis of the signal 's time and frequency properties[10]. In fact, the ability of Fourier analysis to go from time domain to frequency domain and to have a dictionary that allows us to infer properties in one domain from the information in the other domain revolutionized mathematics. But, when computing the Fourier coefficients all values of the function in time domain are taken into account for each frequency and therefore frequency information is obtained at the cost of time information.

This drawback of Fourier transform can be overcome by constructing a complete orthonormal set in $L^2(R)$, the collection of all square integrable functions defined over R of the form $\{\psi_{j,k}\}_{j,k\in Z}$ for some $\psi \in L^2(R)$, known as wavelet system. They are compactly supported and band limited. For this purpose Mallat[11] determined the conditions that a sequence of embedded vector spaces $\{V_j : j \in Z\}$ should satisfy. Under these conditions the sequence will become a multi-resolution analysis (MRA) which in turn yields a compactly supported wavelet system. Since, MRA requires $V_0 \subseteq V_1$ and if $\{\varphi_{1,k} = \sqrt{2}\varphi(2t-k) : k \in Z\}$ is a complete orthonormal set in V_1, for some $\varphi \in V_0$, such that $\int_R \varphi(t)dt = 1$, we can express $\varphi = \sum_{k\in Z} u(k)\varphi_{1,k}$. Letting $\psi(t) = \sum_{k\in Z} v(k)\varphi_{1,k}$ where $v = \{v(k) : v(k) = (-1)^k u(1-k)$ and $W_0 = \left\{\sum_{k\in Z} z(k)\psi_{0,k} : z \in l^2(Z)\right\}$ where $l^2(Z)$ is the collection square summable sequences, we can easily see that $\{\psi(t-k) : k \in Z\}$ is orthonormal to $\{\varphi(t-k) : k \in Z\}$. This W_0 is the complement of V_0 in V_1 and is obviously orthonormal to V_0 and hence $V_1 = V_0 \oplus W_0$. If W_j is the subspace spanned by $\{2^{\frac{j}{2}}\psi(2^j t-k) : k \in Z\}$ then it can be shown that W_j is orthogonal complement of V_j in V_{j+1} for every $\mathrm{j} \in \mathrm{Z}$. Obviously, W_j's are not nested but $\ldots \perp W_0 \perp W_1 \perp \ldots$ As a result $L^2(R) = \oplus_{j\in Z} W_j$ and $V_n = V_m \bigoplus_{j=m+1}^{\infty} W_j$ for $m < n$.

Therefore, $\{2^{\frac{j}{2}}\psi(2^j t-k) : j,k \in Z\}$ forms an orthonormal basis for $L^2(R)$. Thus the MRA eventually yields an orthonormal basis which consists of functions known as wavelets. The projection of any function of $L^2(R)$ in V_j can be thought of as the approximation of that function and the projection over W_j gives the detail to go over to the next approximation level in V_{j+1}. This enables us to view any function of $L^2(R)$ in various resolutions.

B. Variational Mode Decomposition

In this section, we give a brief review of VMD. This technique decomposes the given function f into various components f_k for $k = 1,2,\ldots K$ known as modes using calculus of variation. The Fourier transform of each component of the function $f_k$is assumed to have compact support around the mean of the independent variable of the Fourier transform. This is otherwise known as the central frequency[12] and is denoted byω_k.VMD will find out these central frequencies and the components centered on those frequencies concurrently by minimizing the sum of the lengths of the supports $\Delta\omega_k$ of the Fourier transforms or bandwidths of components subject to the condition that sum of the components is equal to the given function. For this purpose, we construct an analytic function corresponding to the original component $f_k(t)$as$f_k(t) + \mathrm{j} f_k^H(t)$ by finding the Hilbert Transform f_k^H of the component f_k which in fact forms the conjugate harmonic of the component f_k. For the sake of convenience, we wish to shift the bandwidth of the analytic signal $f_k^A(t)$ for k = 1,2,...K, from ω_k to 0. This can be accomplished by multiplying the analytic function f_k^A with the complex exponential $e^{-j\omega_k t}$.Further, we know that the measure of bandwidth of the components can be calculated by finding the integral of the square of the time derivative of the frequency-translated function component

$$f_k^D(t) = f_k^A(t)e^{-j\omega_k t}$$
$$i.e.\, \Delta\omega_k = \int (\partial_t(f_k^D))\overline{(\partial_t(f_k^D))}\, dt$$
$$= \int |\partial_t(f_k^D)|^2 dt$$
$$= \left|\left|\partial_t(f_k^D)\right|\right|_2^2$$

Thus, the bandwidth is the squared L^2 norm of the gradient of the demodulated function components [13].Therefore, the required minimization problem is

$$\underset{f_{k,}\omega_{k,}}{min} \sum_k \left|\left|\partial_t(f_k^D)\right|\right|_2^2 \text{ such that } \sum_k f_k = f$$
$$i.e. \underset{f_{k,}\omega_{k,}}{min} \sum_k \left|\left|\partial_t[((\delta(t) + j/\pi t) * f_k(t))e^{-j\omega_k t}]\right|\right|_2^2$$
$$\text{such that } \sum_k f_k = f$$

The above constrained optimization problem will be solved by converting into unconstrained optimization problem. In fact, we find the central frequencies ω_k for $k = 1,2,\ldots K$and the respective modes by minimizing the sum of the bandwidths using augmented Lagrangian method [14].

$$L(f_k, \omega_k, \lambda) = \alpha \underset{f_k,\omega_k}{min} \sum_k \left|\left|\partial_t(f_k^D)\right|\right|_2^2 + ||f - \sum_k f_k||_2^2 + \lambda(f - \sum_k f_k)$$

where α is the balancing parameter of the data-fidelity constraint and λ is the Lagrangian multiplier. In fact, we can customize our optimization procedure of VMD based on our need.

In the next section, the decomposition of certain test functions are made using VMD and wavelet decomposition techniques. The results are then analyzed by considering various modes under varying bandwidth constraints and noise tolerance of VMD and various levels of different wavelets. Also a comparative study is carried out between the performance of VMD and Wavelet.

3. Experimental Results

Two of the few test functions mainly used by statisticians, namely Doppler, with high frequency oscillations and Bumps, with jump discontinuities, and the sine, a smooth function are considered here for the analysis and comparison of performance between VMD and wavelet decomposition. In fact we study the efficiency of VMD and wavelet decomposition techniques in eliminating noise from the signal and a comparative study is also made. For this purpose we add Gaussian noise to the Doppler and Bumps function respectively. While we work with VMD technique, we have mainly considered three parameters, i.e., number of modes, noise tolerance and bandwidth constraints, which are varied and applied on each function, whose results are obtained using MATLAB. We have obtained results for the number of reconstruction modes 2,3 and 4, for noise tolerance 0,1,2,3 and 4 and for bandwidth constraints 100,90,75,50 and 30 for Doppler whereas for 2000,1000,500,100,90,50 and 30 for Bumps. In wavelet decomposition technique we have used three wavelets, namely coiflet of order 5,4 and 3, for each of which we have varied the approximation levels through 2,3,4,5,6 and 7. The efficiency of noise removal of the above discussed techniques over these test functions made us review the performance of these decomposition techniques over a smooth function as well.

A. Results based on Variational Mode Decomposition

A.1. Results of Doppler

When we used VMD over Doppler, we varied initially the bandwidth constraint$(\alpha)$for modes 2,3 and 4 and for fixed noise tolerance(τ). We used the mean square error (MSE) to study the performance. The conclusion so obtained on the bandwidth constraint show that the resulting MSE is better for $\alpha < 100$. Hence we found out the result by varying bandwidth constraint $\alpha = 90, \alpha = 75, \alpha = 50$ *and* $\alpha = 30$. The best result was found to be for the mode, $k = 2$ at $\alpha = 30$ for fixed noise tolerance $\tau = 0$, which is shown in Table 1. Further, we kept the bandwidth constraint constant at $\alpha = 30$and the noise tolerance was increased which is shown in Table 2. The MSE was found to be negligible for the noise tolerance $\tau = 4$and $k = 2$. The plot of original Doppler function embedded on the first mode of VMD are depicted in Figure 1 for the parameters $= 2$, $\alpha = 30$ and $\tau = 0$ and in Figure 2 for parameters $k = 2$, $\alpha = 30$ and $\tau = 4$.

Table 1. Variation in Bandwidth Constraint

No. of modes	α = 100	α = 90	α = 75	α = 50	α = 30
K=2	0.0852	0.0862	0.072	0.069	0.046
K=3	0.1742	0.1654	0.151	0.0969	0.062
K=4	0.207	0.19623	0.1928	0.1642	0.1466

Table 2. Variation in noise tolerance

No. of modes (k)	τ =1	τ =2	τ =3	τ =4
K=2	0.011	0.012	0.0101	0.010
K=3	0.063	0.065	0.064	0.067
K=4	0.1359	0.1372	0.1352	0.1321

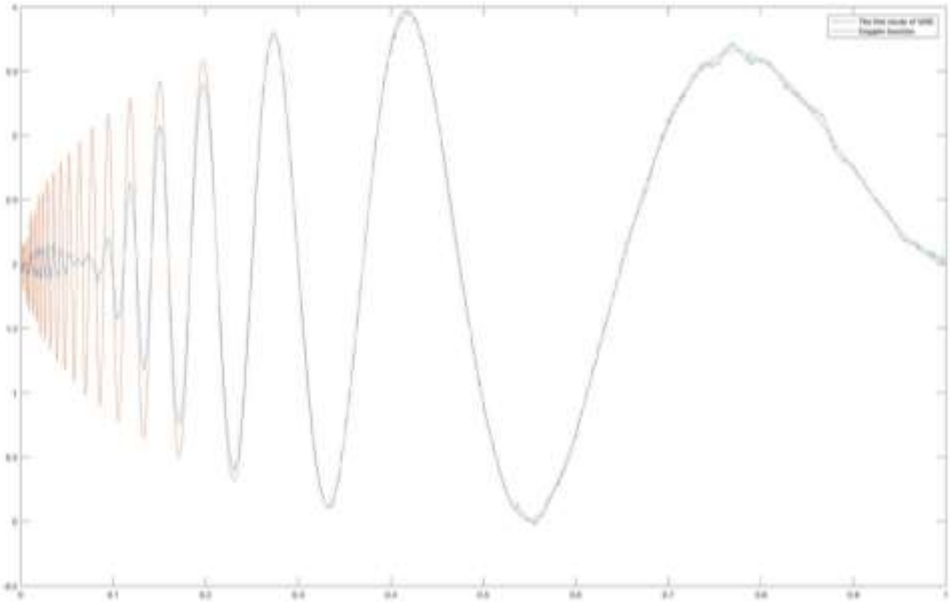

Figure 1

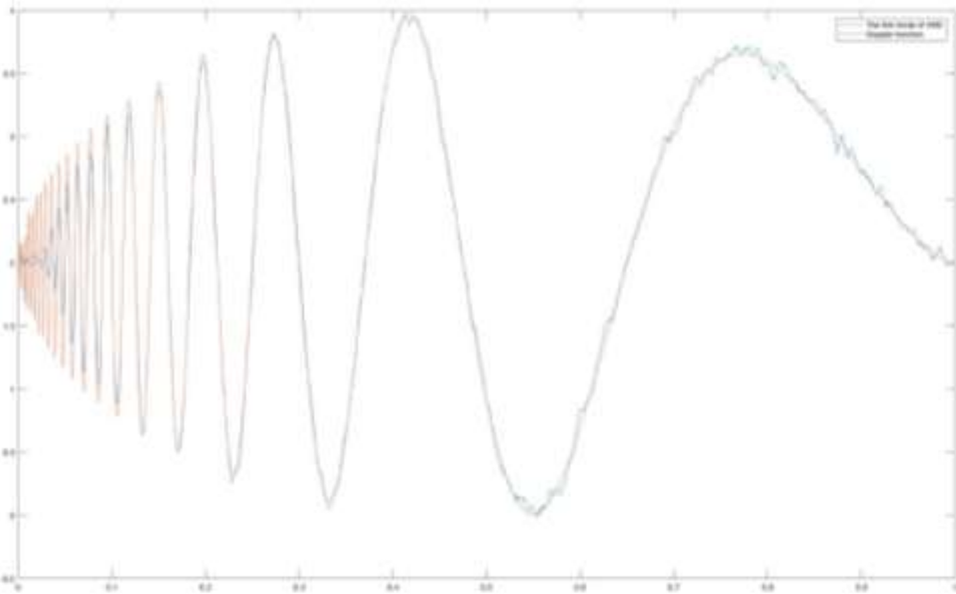

Figure 2

A.1.2 Results of Bumps test function

For Bumps, when the bandwidth constraint α is varied on a scale from 2000 to 30, choosing certain intermediate values, it is seen that the MSE goes on decreasing with decreasing α value, which is shown in Table 3. Thus, the bandwidth constraint is fixed at a value of 30 and the performance is analyzed for different noise tolerance $\tau = 1$, $\tau = 2$, $\tau = 3$ and $\tau = 4$, with number of modes taken as 2,3 and 4 under each case which is shown in Table 4. It is seen that the least MSE occurs for $\tau = 2, k = 2$. The plot of original Bumps function embedded on the first mode of VMD are depicted in Figure 3 for the parameters $= 2$, $\alpha = 30$ and $\tau = 0$ and in Figure 4 for parameters $k = 2$, $\alpha = 30$ and $\tau = 4$.

Table 3. Variation in Bandwidth constraint

No. of modes (k)	$\alpha = 2000$	$\alpha = 1000$	$\alpha = 500$	$\alpha = 100$	$\alpha = 90$	$\alpha = 50$	$\alpha = 30$
$K = 2$	0.1795	0.1648	0.1577	0.1159	0.1045	0.0811	0.0668
$K = 3$	0.1858	0.1779	0.1739	0.1529	0.1563	0.1392	0.1356
$K = 4$	0.1886	0.1854	0.1813	0.1689	0.1669	0.1656	0.1596

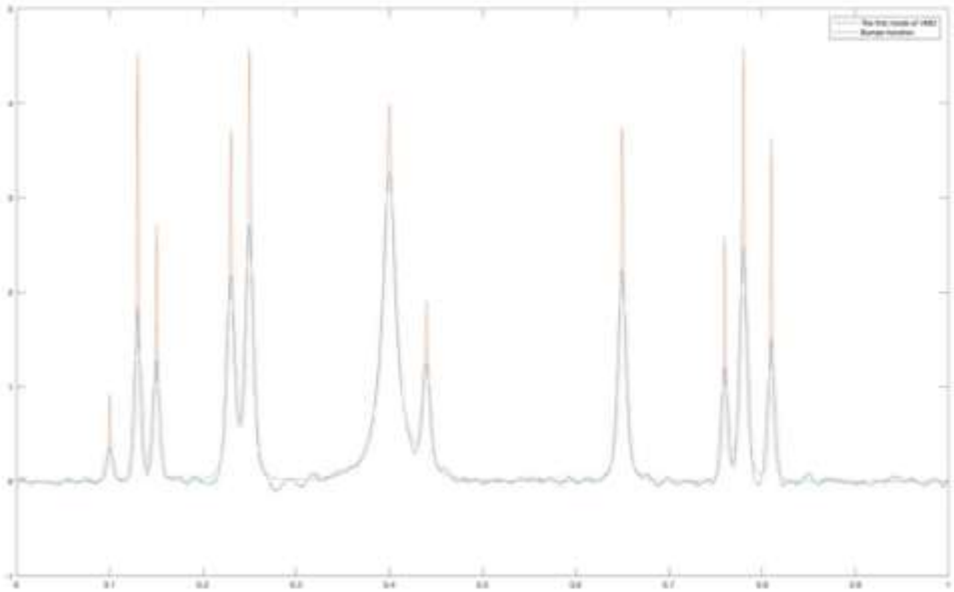

Figure 3.

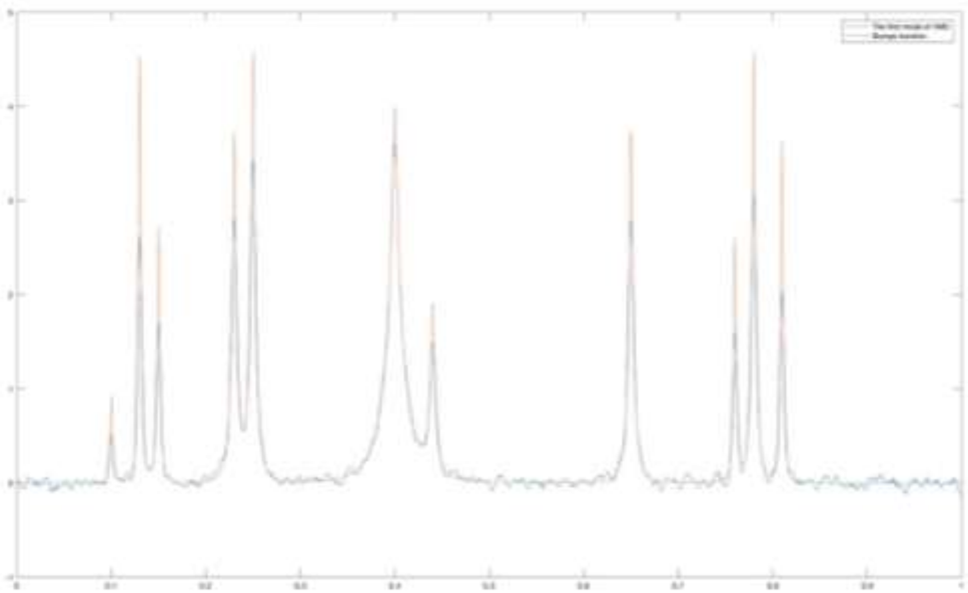

Figure 4

Table 4. Variation in noise tolerance

No. of modes (k)	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$
$K = 2$	0.0244	0.0237	0.024108	0.025745
$K = 3$	0.103487	0.0936	0.0842	0.086071
$K = 4$	0.1492	0.1486	0.150116	0.150646

B. Results based on Wavelets

When wavelet techniques are used over the considered test functions, the decompositions at various approximation levels are obtained using coiflet of order 3,4 and 5 and analyzed. For both Doppler and Bumps, we see that the MSE is the least for second level, under coiflet of all orders analyzed, i.e., lower the approximation level, lower the MSE. It is also observed that among coiflet of order 3,4 and 5, coiflet of order 5 yields better results than the other ones. The plot of original Doppler function embedded on the seventh and second approximation level of coiflet wavelet of order 5 are depicted in Figure 5 and in Figure 6, for the plot of original Bumps function embedded on the seventh and second approximation level of coiflet wavelet of order 5.

B.1 Wavelet results of Doppler

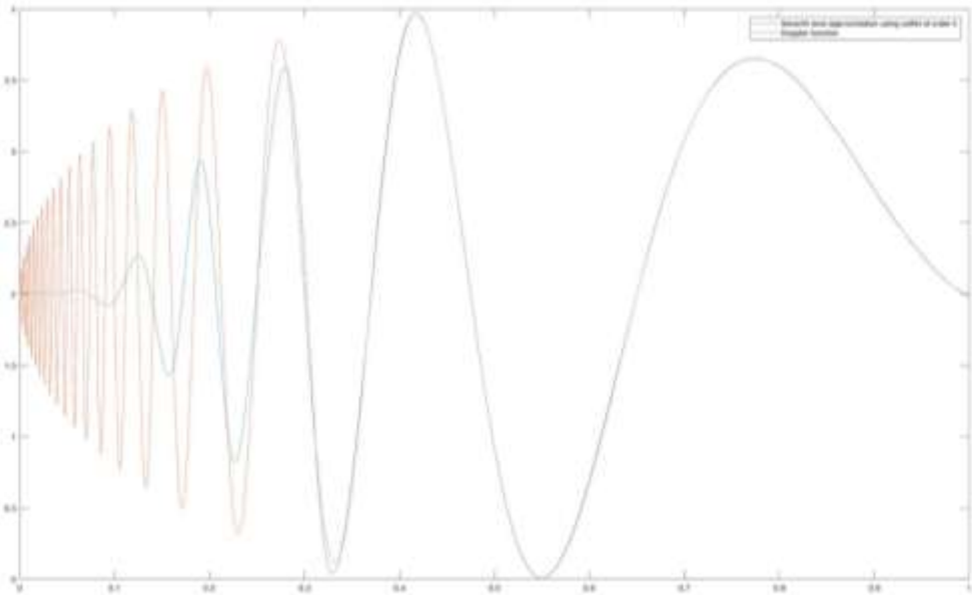

Figure 5(a)

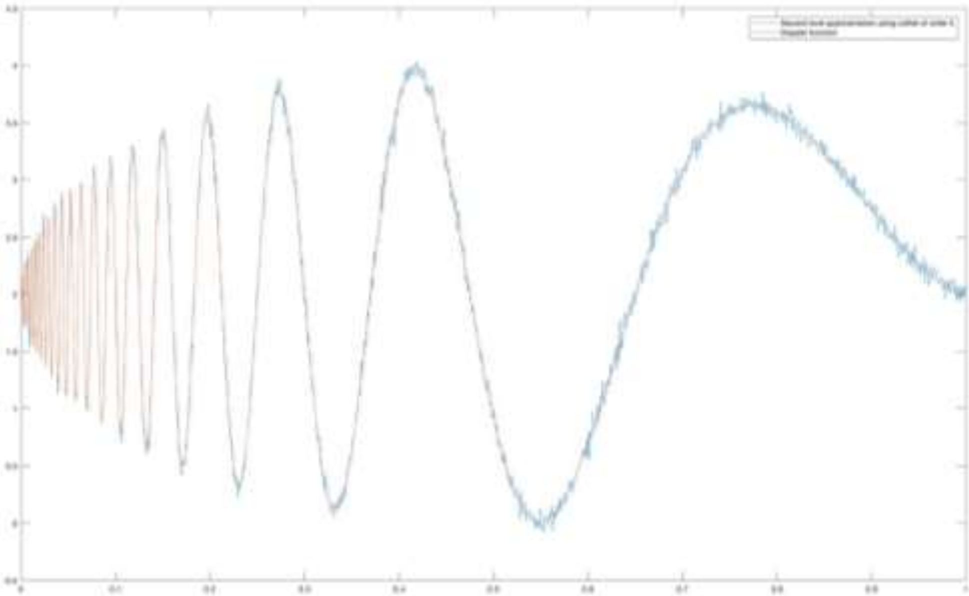

Figure 5(b)

B.2 Wavelet results of Bumps

Table 6. Variation in levels of various wavelets for bumps function

Wavelets	Level 7	Level 6	Level 5	Level 4	Level 3	Level2
Coif5	0.2313	0.1617	0.0629	0.0213	0.0063	0.0034
COif4	0.2308	0.1618	0.0671	0.0251	0.0053	0.0032
Coif3	0.2293	0.1620	0.0694	0.0228	0.0067	0.0034

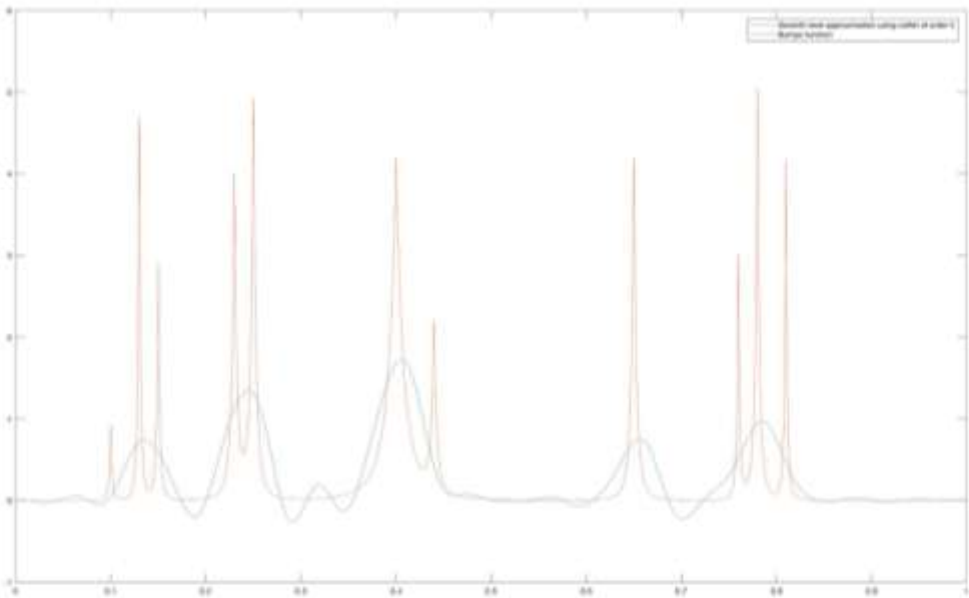

Figure. 6(a)

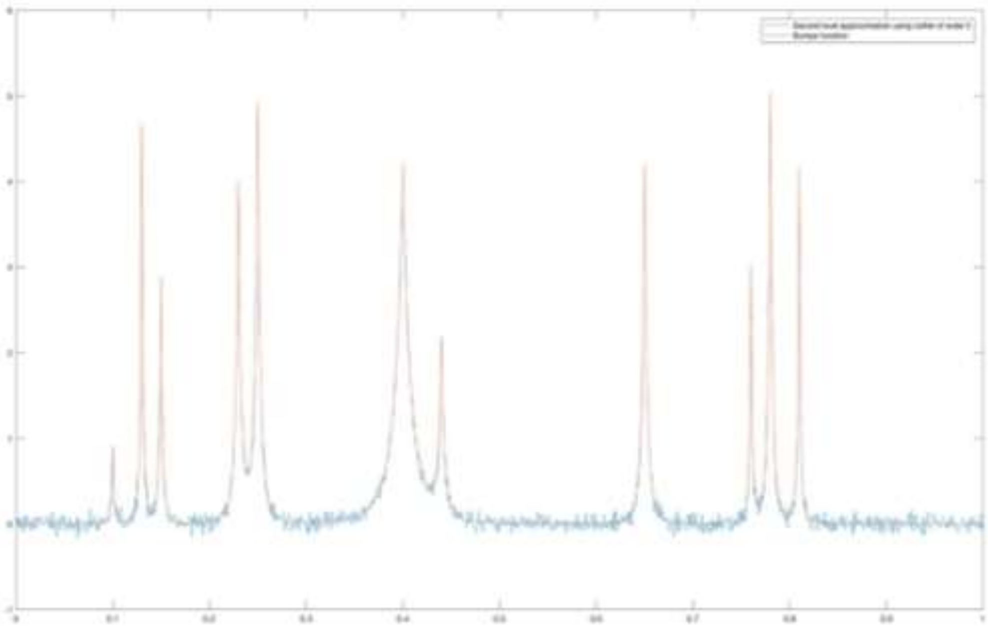

Figure. 6(b)

C. Wavelet and VMD for a smooth curve

Because of the difference in the nature of smoothness of the test functions namely Doppler and Bumps, we got motivated to study the performance of the above discussed decomposition techniques also over a smooth function. Hence, we considered $f(x) = sinx$ for the analysis of Wavelet and VMD techniques in order to conclude their performances over smooth and non-smooth functions.

Table 7. Variation in Bandwidth constraint and noise tolerance of sine function

No. of modes(K)	$\alpha = 2000$ $\tau = 0$	$\alpha = 1000$ $\tau = 0$	$\alpha = 500$ $\tau = 0$	$\alpha = 100$ $\tau = 0$	$\alpha = 90$ $\tau = 0$	$\alpha = 50$ $\tau = 0$	$\alpha = 30$ $\tau = 4$
$K = 2$	0.001847	0.000639	0.000231	0.000022	0.000013	0.000004	0.000001
$K = 3$	0.002137	0.000736	0.000265	0.000025	0.000020	0.000001	0.000000
$K = 4$	0.002522	0.000895	0.000322	0.000031	0.000026	0.000001	0.000000

Table 8. Variation in levels of various wavelets for sin function

Wavelets	Level 2	Level 4	Level 5	Level 6	Level 8	Level 10
Coif5	0.5516	0.5304	0.4875	0.3887	0.2615	0.098
COif4	0.5516	0.5264	0.4834	0.388	0.2585	0.143
Coif3	0.5532	0.5289	0.4814	0.3882	0.2548	0.1454

When the VMD technique is applied over sine function, the MSE decreases as the bandwidth constraint decreases and an increase in noise tolerance results in an decreasing MSE. The least error is obtained at the same case as obtained for the specific test functions Doppler and Bumps. Further, in the case of wavelet technique the MSE decreases when the approximation level increases for the sine function and it is found that MSE obtained by varying the approximation levels of various wavelets is comparatively more than the MSE obtained by VMD. Also it is interesting to note that the MSE is less for higher approximation levels in case of smooth function whereas the MSE is found to be less for lower approximation levels in case of specific test functions.

4. Conclusion

We performed the VMD and Wavelet based noise removal over spatially inhomogeneous test functions, Doppler with high frequency oscillations and Bumps with jump discontinuities. Also, we have performed the wavelet and VMD based noise removal on sine function to observe their performances on a spatially homogeneous function and thereby we extended our study to spatially homogeneous functions as well in order to compare the performance of these techniques with spatially homogeneous or inhomogeneous functions. Our study reveals that the VMD technique yields better results when the parameters namely the number of modes $k = 2$, bandwidth constraint $\alpha = 30$and noise tolerance $\tau = 4$, despite the spatial characteristics of the function. Thus the VMD technique performs equally well for all the functions that we studied, irrespective of the spatial inhomogeneity and thus with or without prior knowledge for the afore mentioned parameters. On the other hand, while we used linear wavelet techniques, the best results for Doppler and Bumps were obtained at level 2 whereas for the sine function, it was obtained at level 10. Thus, the approximation levels of linear wavelet techniques are varied to get better results depending upon the spatial characteristics of the function.

From the above discussion, we note that the VMD technique performs on par with linear wavelet techniques for the spatially homogeneous functions and VMD performs better than the linear wavelet technique in the case of spatially inhomogeneous functions. Also, as better results are obtained in VMD technique for the same parameters despite the spatial characteristics of the signal, the prior knowledge of the signal is not required. Hence our comparative study concludes based on the efficiency of noise removal on these functions that in the case of signals without prior knowledge or when the signal is spatially inhomogeneous, VMD technique will result in better recovery of the signal than the linear wavelet technique after the noise removal.

5. References

[1]. Childers, D. G., Varga, R. S. and Perry.N. W. (1970). Composite Signal Decomposition. *IEEE Transactions on Audio and Electroacoustics*. vol. AU-18, No. 4: 471-477.

[2]. Palanisamy. T., (2016). Smoothing the Difference-based Estimates of Variance UsingVariational Mode Decomposition.*Communications in Statistics - Simulation and Computation ,(In Press)*

[3]. R.Todd Ogden. (1997). Essential wavelets for Statistical Applications and Data Analysis. Birkhäuser.

[4]. Dragomiretskiy, K. and Zosso, D. (2014). Variational Mode Decomposition. *IEEE Transactions on Signal Processing*. 2:531-544.

[5]. Donoho, D. L. and Johnstone I. M \(1994). Ideal Spatial Adaptation by Wavelet Shrinkage.*Biometrika,* Vol. 81, No. 3: 425-455.

[6]. Sadhukhan, D. and Mitra, M. (2014). ECG Noise Reduction Using Linear and Non-linear Wavelet Filtering.*Proc. of Int. Conf. on Computing, Communication & Manufacturing*. 2014

[7]. Amato U and Vuza D.T. (1994). Wavelet Regularization for Smoothing Data. *Technical report N. 108/94*, *Instituto per ApplicazionidellaMatematica*, *Napoli*.

[8]. Antoniadis, A. (1994). Smoothing noisy data with coiflets.*StatisticaSinica*4 (2),651-678.

[9]. Antoniadis, A. and Gregoire, G. and Mckeague, I. (1994). Wavelet methods for curve estimation. *J. Amer. Statist. Assoc.* 89 (428), 1340-1353.

[10]. Palanisamy, T. and Ravichandran, J. (2015). A wavelet-based hybrid approach to estimate variance function in heteroscedastic regression models. *Stat. Papers.* 56(3):911-932.

[11]. Mallat, S. G. (1989). Multiresolution approximations and wavelet orthonormal bases of L^2 (R). *Transactions of the American Mathematical Society*.Vol.315, No.1, 69-87

[12]. Aneesh C, S. K, Hisham P M, Soman, K. P. (2015). Performance comparison of Variational Mode Decomposition over Empirical Wavelet Transform for the classification

of power quality disturbances using Support Vector Machine.*Procedia Computer Science*.46:372-380.

[13]. Wang, Y. M. R, Xiang, J. and Zheng, W. (2015). Research on variational mode decomposition and its application in detecting rub-impact fault of the rotor system. *Mechanical Systems and SignalProcessing*.60-61:243-251.

[14]. Huang, N. E., Shen, Z., Long, S. R., Wu, M.. C., Shih, H. H., Zheng, Q., Yen,N. C., Tung, C. C. and.Liu. H. H. (1998). The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis.*Proc. Royal Soc. A: Math., Phys. Eng. Sci*.. vol. 454, no. 1971:. 903–995.

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